Note on Network Game

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1 Strategic Network Formation

In this section, we not only have predictions about which networks might form (i.e. pairwise stability), but we also have measures of which nemtworks are "best" from society's point of view (i.e. efficiency). Some graphical models in this section also provide answers to why networks take particular forms. To achieve this goal, given a network $\{N, G\}$, we introduce utility function of each players.

Definition 1.1 (Utility function)

The payoff to a player i is represented by a function $u_i : G(N) \to R$.

In these models, externalities refer to situations where the utility to one individual are affected by the actions of others, where those actions do not directly involve the individual in question. For example, distance-based utility brings positive externalities, as added links can only bring players closer together, while coauthor model brings negative externalities, where an individual prefers his or her neighbors have fewer connections rather than more.

1.1 Pairwise Stability

Definition 1.2 (Pairwise stability (equilibrium))

A network G is pairwise stable if no player wants to sever a link and no two players both want to add a link, i.e.

1. $u_i(G + ij) \le u_i(G)$ *or* $u_j(G + ij) \le u_j(G)$*, at least one strictly, for all* $ij \notin G$;
2. $u_i(G - ij) \le u_i(G)$ *and* $u_j(G - ij) \le u_j(G)$ *for all* $ij \in G$.

Note on *This definition is used to capture the fact that forming a relationship or link between two players usually involves mutual consent, while severing a relationship only involves the consent of one player. Specifically, in a PS G,*

1. *No two agents would both benefit by adding a link between themselves;*
2. *No agent want to delete some links that it is directly involved in.*

Actually, the second requirement, neither player has an incentive to change his or her action, forms a Nash equilibrium, though this alone does not make much sense in a social setting.

Note on *Limitations*

1. *Pairwise stability is a weak notion in that it only considers deviations on a single link at a time;*
2. *Pairwise stability considers only deviations by at most a pair of players at a time.*

Lemma 1.1

Any pairwise stable network has at most one component.

Proof

■

1.2 Efficiency

Definition 1.3 (Efficiency)

A network g is efficient relative to a profile of utility functions $(u_1, ..., u_n)$ if g maximizes the utilitarian welfare, i.e. $\sum_i u_i(g) \geq \sum_i u_i(g')$ for all $g' \in G(N)$.

Note on *For example, if $n = 3$, let $W(G)$ denote the maximized utilitarian welfare, i.e. $G \in \operatorname{argmax} \sum_{i \in N} u_i(G)$:*

1. *Empty network: $W(G) = 0$;*
2. *Complete network: $W(G) = 3(2\delta - 2c)$;*
3. *Star network: $W(G) = 2\left(\delta + \delta^2 - c\right) + 2\delta - 2c$.*

Definition 1.4 (Pareto Efficiency)

A network g is pareto efficient relative to a profile of utility functions $(u_1, ..., u_n)$ if there does not exist any $g' \in G$ such that $u_i(g') \geq u_i(g)$ for all i with strict inequality for some i.

Note on *If g is efficient relative to $(u_1, ..., u_n)$ then it must also be Pareto efficient relative to $(u_1, ..., u_n)$. However, the converse is not true.*

1.3 Distance-Based Utility: Symmetric Connection Model with Positive Externalities (Jackson & Wolinsky, 1996)

Note that symmetric connection model is a special case of distance-based utility, in this section, we focus on this special case. There are n players and friendship among players. Each player receives positive benefit from direct and indirect friendships, where the utlity depends only on the distance, and keeping friendship is costly.

Definition 1.5 (Symmetric Connection Model: utility funciton)

$$u_i(G) = \sum_{j \neq i} \delta^{l(i,j)} - d_i(G) * c,$$

where $\delta \in (0,1)$ is a decay parameter, $l(i,j)$ is the length of a shortest path between i and j, and c is the cost of keeping a friendship.

Note on *For example, $l(i,j) = +\infty$ if i and j are not connected and $\delta^{l(i,j)} = 0$, the utility from a friend is $\delta - c$, the utility from a friend of friend is δ^2.*

Lemma 1.2 (Symmetric Connection Model: efficiency)

1. *If $\delta + \frac{(n-2)\delta^2}{2} < c$, empty network is a unique efficient network;*
2. *If $\delta - \delta^2 < c < \delta + \frac{(n-2)\delta^2}{2}$, star network is a unique efficient network;*
3. *If $\delta - \delta^2 > c$, complete network is a unique efficient network.*

Proof

1. If $\delta + \frac{(n-2)\delta^2}{2} < c$, no one wants to add a link.
2. If $\delta - \delta^2 < c < \delta + \frac{(n-2)\delta^2}{2}$, a star involves the minimum number of links needed to ensure connectness, and it has each player within two links of every other player. Note that all connected networks with $n-1$ links has the same cost and payoffs from direct connections, but the star network with the property that connects each node within two links presents unique efficiency.
3. If $\delta - \delta^2 > c$, then adding a link between any two agents i and j will always increase total welfare. This follows because i and j may enjoy at most δ^2 from any sort of indirect connection.

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Theorem 1.1 (Symmetric Connection Model: pairwise stability)

1. *If $\delta < c$, in any pairwise stable network, $d_i(G) = 0$ or $d_i(G) \geq 2$;*
2. *If $\delta - \delta^2 < c < \delta$, star network is pairwise stable but may not be unique;*
3. *If $\delta - \delta^2 > c$, complete network is a unique pairwise stable network.*

Proof

1. If $\delta - \delta^2 < c < \delta$, for star network, adding a link is not beneficial ($\delta - \delta^2 < c$), deleting a link is also not beneficial ($c < \delta$). If $c < \delta - \delta^3$, then "circle" network would also be pairwise stable when $n = 4$.
2. If $\delta - \delta^2 > c$, adding links is beneficial to all nodes.

■

Lemma 1.3

Consider a variation on the symmetric connections model, where $\delta^{l(i,j)} = b(l(i,j))$, and $b(k)$ decreasing in k and being positive as long as the distance $K > 1$, i.e. $c > b(1) > b(2) > b(3) > \cdots > b(k) > b(k+1) > \cdots b(K) > 0$.

Then as n grows, the diameter of any nontrivial pairwise stable network is bounded by $2K+1$.

Proof Suppose not, there exists i and j with distance at least $2K+2$. Then for any node $h \in N_i^1(g)$, i.e., i's neighbors, according to pairwise stability, we have

$$-c + \sum_{h' \in N_i^K(g)} \left[b\left(\ell_g\left(h, h'\right)\right) - b\left(\ell_{g-hi}\left(h, h'\right)\right)\right] \geq 0.$$

That is, for $h \in N_i^1(g)$, their utility of maintaining this link is greater than or equal to deleting this link. Note that j must be at a distance of more than $K+1$ in g from every $h' \in N_i^K(g)$, otherwise we can find a shorter path and the shortest distance between i and j is smaller than $2K+2$. Similarly, j must be at a distance of more than $2K$ in g from every $h \in N_i^1(g)$. Then, the utility of adding a link between i and j is

$$-c + b(2) + \sum_{h' \in S} b\left(\ell_g\left(h, h'\right)\right) > -c + \sum_{h' \in N_i^K(g)} \left[b\left(\ell_g\left(h, h'\right)\right) - b\left(\ell_{g-hi}\left(h, h'\right)\right)\right] \geq 0.$$

Thus this is not pairwise stable, and contradicts our assumption. ■

1.4 Price of Anarchy and Price of Stability

To capture the degree of inefficiency in stable networks, we introduce the definition of price of anarchy and price of stability and study by the following example. Consider a specific distance-based utility model: $u_i(G) = \sum_{j \neq i} -l(i,j) - d_i(G) * c$, where utilities are always negative, and can be interpreted as a cost of communication. Let $C(G) = |W(G)|$ denote the total communication cost under network G.

Definition 1.6 (Price of Anarchy)

$$\frac{\max_{G \in PS} C(G)}{C\left(G_E\right)}$$

Note on *Here 1 means all PS networks are efficient, while <1 means some PS networks are inefficient.*

Definition 1.7 (Price of stability)

$$\frac{\min_{G \in PS} C(G)}{C\left(G_E\right)}$$

Note on *Here 1 means that all efficient networks will be stable (there may be some stable inefficient networks, i.e. price of anarchy <1), while <1 means that all stable networks are inefficient.*

Lemma 1.4 (Efficient Networks)

1. *Efficient network must be connected;*
2. *If $c < 1$, the complete network is a unique efficient network;*
3. *If $c > 1$, the star network is a unique efficient network.*

Note on *If $c < 1$, the unique pairwise stable network is complete network, and so the price of anarchy and the price of stability are both one, otherwise the star network is pairwise stable, and so the price of stability remains 1, but there are other pairwise stable netowrks and so the price of anarchy decreases to below 1.*

Lemma 1.5 (Fabrikant)

1. *The diameter of any pairwise stable networks is at most $2\sqrt{c} + 1$.*
2. *Any pairwise stable networks has at most $n - 1 + 3n^2/\sqrt{c}$ links.*
3. *The price of anarchy is at most $17\sqrt{c}$.*
4. $\max_{G \in PS} C(G) \leq n(n-1)(2\sqrt{c}+1) + 2(n-1)c + 6n^2\sqrt{c}$.

1.5 Coauthor Model with Negative Externalities (Jackson & Wolinsky)

Definition 1.8 (Coauthor Model: utility function)

Player i's utility function under graph G is $u_i(G)$, where $d_i(G)$ is the number of research projects that i is involved, and the costs of link formation is implicit.

$$u_i(G) = \begin{cases} \sum_{j:ij \in G} \left(\frac{1}{d_i(G)} + \frac{1}{d_j(G)} + \frac{1}{d_i(G)d_j(G)} \right) & \text{if } d_i(G) > 0 \\ 1 & \text{if } d_i(G) = 0 \end{cases}$$

Note on *This utility function also reveals that you do not want your coauthor to do a project without your, i.e. $u_i(G + jk) \leq u_i(G)$.*

Theorem 1.2

1. *If n is even, efficient networks consist of $\frac{n}{2}$ separate pairs;*
2. *When n is even and $n \geq 4$, PS networks are inefficient, and can be partitioned into complete components, and each of the components has a different number of nodes.*

Note on *Inefficiency comes from the fact that players ignore the negative externalities: if i and j form a link, they will spend less time on projects with other coauthors.*

Proof

■

1.6 Small Worlds: Islands-connection Model

In reality, there are relatively small diameters (largest distance between two nodes) and average path length, and the real world has relatively high clustering, and this model can be used to capture this characteristic.

Definition 1.9 (Islands-connection Model: utility function)

Player i's utility function under graph G is $u_i(G)$, where $n = KJ$, K is the number of islands and each island has J players, D is the truncation, if the minimum path length between two players is more than D links, then they do not receive any value from each other. $c_{ij} = c$ if i and j are in the same islands and $c_{ij} = C$ otherwise, where $C > c$.

$$u_i(G) = \sum_{j \neq i: l(i,j) < D} \delta^{l(i,j)} - \sum_{j: ij \in G} c_{ij}$$

Theorem 1.3

If $c < \delta - \delta^2$ and $C < \delta + (J-1)\delta^2$, then any pairwise stable or efficient networks is such that

1. *Players on each island forms a complete component;*
2. *The diameter and average path length are no greater than $D + 1$;*
3. *If $\delta - \delta^3 < C$, then a lower-bound on clustering is $(J-1)(J-2)/J^2K^2$.*

Note on *Intuition*

1. *Low costs of connections to nearby players (those on the same island) lead to high clustering.*
2. *The high value to linking to other islands (accessing many other players) leads to low average path length.*
3. *The high cost to linking across islands means that there are only a few links across islands.*

Proof

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Definition 1.10 (Shortcut link)

Shortcut links are those which link distant parts of the network and if deleted would substantially alter the distance between the connected nodes.

1.7 Link stability and efficiency

Since efficiency and stability may not coincide, we want to do transfers among the players to solve this problem, e.g. taxing and subsidizing links.

To be continue

Definition 1.11 (Transfer rule)

A (balanced) transfer rule is a function $t : G \to R^n$ such that $\sum_{i \in N} t_i(G) = 0$. Given the transfer rule t, i's net payoff becomes $u_i(G) + t_i(G)$.

Definition 1.12 (Egalitarian transfer rule)

A transfer rule t is called an egalitarian transfer rule if $u_i(G) + t_i(G) = \frac{W(G)}{n}$.

Note on *Given egalitarian transfer rule, players' payoff is completely equalized, and any efficient network maximizes each individual's net payoff, thus is pairwise stable.*

Definition 1.13 (Component balanced)

A transfer rule t is component balanced if $\sum_{i\in S} t_i(G) = 0$ for any components S in G.

Note on *Component balanced means that there are not much payoff externalitites across components.*

Definition 1.14 (Component-decomposable)

A profile of utility function is component-decomposable if for all $i \in N$, $u_i(G) = u_i\left(G|_{S_i}\right)$ where S_i is a component that contains i.

Definition 1.15

Let G^{ij} denote the network that switches the positions of i and j, e.g. $g_{kh} = g^{ij}_{kh}$, $g_{jk} = g^{ij}_{ik}$ and $g_{ik} = g^{ij}_{jk}$.

Definition 1.16

i and j are complete equals relative to u and G if

1. *$ik \in G$ iff $jk \in G$ for all $k \neq i, j$;*
2. *$u_k(G) = u_k\left(G^{ij}\right)$ for all $k \neq i, j$ and G;*
3. *$u_i(G) = u_j\left(G^{ij}\right)$ and $u_j(G) = u_i\left(G^{ij}\right)$ for all G.*

Definition 1.17 (Equal treatments of equals)

A transfer rule t satisfies equal treatments of equals if $t_i(G) = t_j(G)$ when i and j are complete equals relative to u and G.

Theorem 1.4 (Incompatibility of Pairwise Stability and Efficiency)

There exist component-decomposable utility functions such that every pairwise stable network relative to any component-balanced transfer rule satisfying equal treatment of equals is inefficient.

2 Games on Networks

In this section, we discuss basic graphical games, in which the binary graphical game are the most famous.

2.1 Pure Binary Graphical Games

We assume this is a n player simultaneous move game with binary actions, and a network (N, G) is exogenously given. Agent i's utility function depends on both i's action and its neighbors' actions, i.e. $u_i\left(x_i, x_{N_i(G)}\right)$, where $x_i \in \{0, 1\}$ and $x_{N_i(G)}$ is a profile of i's neighbors' actions under G.

Definition 2.1 (PSNE)

A strategy profile x is a pure strategy Nash equilibrium if

1. $u_i\left(1, x_{N_i(G)}\right) \geq u_i\left(0, x_{N_i(G)}\right)$ *if* $x_i = 1$;
2. $u_i\left(0, x_{N_i(G)}\right) \geq u_i\left(1, x_{N_i(G)}\right)$ *if* $x_i = 0$.

Note on *Nonexistence of PSNE There are also games without PSNE. For example, there are two types of players, either"conformists" or "rebels". Conformists are willing to match their actions with the majority of their neighbors, while rebels are willing to match their actions with the minority of their neighbors. One conformist and one rebel dyad does not have PSNE, though mixed-strategy equilibria are naturally defined and always exist.*

2.1.1 Threshold Games of Complements

Definition 2.2 (Utility function)

$u_i\left(1, x_{N_i(G)}\right) \geq u_i\left(0, x_{N_i(G)}\right)$ *iff* $\sum_{j \in N_i(G)} x_j \geq t_i$, *where* t_i *is a threshold.*

Note on *The more neighbors take action 1, the more incentive to take action 1.*

Note on *Special case (costly technology adoption)*

$$
\begin{aligned}
u_i\left(1, x_{N_i(G)}\right) &= a_i\left(\sum_{j \in N_i(G)} x_j\right) - c_i \quad a_i, c_i > 0 \\
u_i\left(0, x_{N_i(G)}\right) &= 0 \\
t_i &= \frac{c_i}{a_i}
\end{aligned}
$$

2.1.2 A "Best-Shot" public goods game

Definition 2.3 (Utility function)

$$
\begin{aligned}
u_i\left(1, x_{N_i(G)}\right) &= 1 - c \quad 1 > c > 0 \\
u_i\left(0, x_{N_i(G)}\right) &= \begin{cases} 1 \text{ if } x_j = 1 \text{ for some } j \in N_i(G) \\ 0 \text{ if } x_j = 0 \text{ for all } j \in N_i(G) \end{cases}
\end{aligned}
$$

Note on *If one of your neighbors learn information, you can have that information without any cost. Here agents have incentive for free-riding.*

Lemma 2.1

In any pure strategy equilibrium,

1. *If a player chooses action 1, none of her neighbors choose action 1.*
2. *Any player who chooses action 0 has at least one neighbor who chooses action 1.*

Lemma 2.2

Consider the "best-shot" public goods game. In any pure strategy equilibria, the players who choose action 1 form a maximal independent set.

Lemma 2.3

Consider the "best-shot" public goods game. Any pure strategy equilibrium x of $G + ij$ satisfies either x is an equilibrium of G or there exists an equilibrium under G in which a strict superset of players chooses action 1.

Proof If $G + ij = G$, then obviously x is still an equilibrium under G. Otherwise, take any equilibrium x' of $G + ij$ and let $A = \{i \in N : x_i = 1\}$, then A is a maximal independent set under $G + ij$. If A is also a maximal independent set under G, then x is also an equilibrium of G. If not, then A is still an independent set under G, thus there is a strict superset of A that is a maximal independent set under G. ■

2.2 Semi-Anonymous Binary Graphical Games

Here players only care about how many of their neighbors take action 0 and 1, but do not care about which of their neighbors take action 0 and 1. The utility function is $u_{d_i}(x_i, m)$, where m is the number of players in $N_i(G)$ taking action 1.

Definition 2.4 (Complements)

A semi-anonymous graphical game shows (strict) strategic complements if $u_d(1, m) - u_d(0, m) \geq u_d(1, m') - u_d(0, m')$ for all d and $(m > m')$ $m \geq m'$.

Note on *Threshold There is a threshold $t(d)$ such that if more than $t(d)$ neighbors choose action 1, then the node with degree d prefers action 1, while if fewer than $t(d)$ neighbors choose action 1, then the node with degree d prefers action 0.*

Lemma 2.4

Pure strategy equilibria always exist. The set of pure strategy equilibria forms a complete lattice and there exists a maximum equilibrium where each player's action is at least as high as in every other equilibrium, and exists a minimum equilibrium

Proof

■

Lemma 2.5

Consider a semi-anonymous graphical game of strategic complements such that $t(d+1) \leq t(d)$ for each d. If we add links to G and have a denser network $G' \supseteq G$, for any equilibrium x of G, there exists an equilibrium x' of G' such that all players play at least as high an action under x' as under x.

Proof Suppose we add some links to G and we have G', then

1. Case 1, for links to node with $x_i = 1$ and node with $x_j = 1$, there is no impact, since larger d only creates incentive to choose 1.
2. Case 2, for links to node with $x_i = 1$ and node with $x_j = 0$, there is no impact for node i, while j may choose 1 or 0, in either case, $x'_j \geq x_j$.
3. Case 3, for links to node with $x_i = 0$ and node with $x_j = 0$, a larger d creates incentive to choose 1, but in either case, $x' \geq x$.

■

Definition 2.5 (Substitutes)

A semi-anonymous graphical game shows (strict) strategic substitutes if $u_d(1,m) - u_d(0,m) \leq u_d(1,m') - u_d(0,m')$ for all d and $(m > m')$ $m \geq m'$.

Note on *Marginal incentive for taking action 1 is increasing in m in the case of complements and decreasing in the case of substitutes.*

Note on *Threshold There is a threshold $t(d)$ such that if more than $t(d)$ neighbors choose action 1, then the node with degree d prefers action 0, while if fewer than $t(d)$ neighbors choose action 1, then the node with degree d prefers action 1.*

Note on *Pure strategy equilibria may not exist. The set of pure strategy equilibria does not necessarily form a complete lattice.*

2.2.1 A "Couples" Game

Definition 2.6 (Utility function)

A player strictly prefers action 1 iff at least one neighbor takes action 1.

$$
\begin{aligned}
u_{d_i}(1,m) &= & 1-c \quad m \geq 1 \\
u_{d_i}(1,0) &= & -c \\
u_{d_i}(0,m) &= & 0
\end{aligned}
$$

Note on *For example, learning a specific skill is painful but also enjoyable when there is at least one friend to practice it with, e.g. playing tennis, some video games.*

2.2.2 A coordination game

Definition 2.7 (Utility function)

A player prefers to coordinate her action with neighbors, and her payoff is related to the fraction of neighbors who play the same action.

$$u_{d_i}(1,m) = a\frac{m}{d_i}$$
$$u_{d_i}(0,m) = b\frac{d_i - m}{d_i}$$

2.3 Quadratic Payoffs

Let $x_i \in [0,+\infty)$, and $u_i(x_i,x_{-i}) = x_i - \frac{1}{2}x_i^2 + \delta\sum_{j\neq i} g_{ij}x_ix_j$, where $0<\delta<1$. Here $g_{ij}>0$ means strategic complementarity between i and j, while $g_{ij}<0$ means strategic substitutability between i and j, and u_i is strictly concave in x_i.

Definition 2.8 (Bonacich centrality vector)

$\boldsymbol{x} = (\mathbf{I} - \delta\boldsymbol{G})^{-1}1$

Note on *FOC*

$$x_i = 1 + \delta\sum_{j\neq i} g_{ij}x_j \quad x = 1 + \delta Gx$$

Actually through FOC, given that $(\mathbf{I} - \delta\boldsymbol{G})$ *is invertible and the solutions are nonnegative (i.e. G is sufficiently "small"), then we have* $\boldsymbol{x} = (\mathbf{I} - \delta\boldsymbol{G})^{-1}1$.

Note on *Also,* $\boldsymbol{x} = \mathbf{1} + \delta \text{G} + \delta^2 G^2 + \cdots = \sum_{k=0}^{\infty}\delta^k G^k$.[1] *When* $g_{ij}\in\{0,1\}$, x_i *is the sum of the number of walks starting from i, where a length-k walk is discounted by* δ^k.

Note on *Example*

Note on *Interpretation The idea is that node i is considered to be central if i's neighbors are also central.* x_i *is high when i has many friends (i's degree), if i's friends have many friends (length-2 walk), if friends of i's friends have many friends (length-3 walk), ...*

Note on *More Bonacich central player chooses a larger equilibrium action.*

Lemma 2.6

When $G\geq 0$, *adding a link between* j *and* k *increases* i*'s equilibrium action iff* i *is path-connected to either* j *or* k.

Note on *Example of cournot competition Consider a Cournot competition with linear inverse demand function and constant marginal cost, here* ρ_{ij} *denotes the demand substitutability or*

[1] Theorem, Inverse of Matrix is the Sum of Power

complementarity between two products.

$$u_i(x_i, x_{-i}) = \left(a - x_i - \sum_{j \neq i} \rho_{ij} x_j\right) x_i - c x_i$$
$$= (a-c)x_i - x_i^2 - \sum_{j \neq i} \rho_{ij} x_i x_j$$

2.4 Bayesian Network Games (Galeotti et al. 2010)

In a n player semi-anonymous graphical game, the utility function is $u_{d_i}(x_i, m)$, where m is the number of players in $N_i(G)$ taking action 1. Players know partially about the network, know their own degree but have a belief about their neighbors' degrees. This thus is a bayesian game where each player's type is his or her degree. Let $P(d)$ be the common degree distribution of each other, and we assume players' degrees are mutually independent, $P(0) = 0$ and full support.

Let $\sigma(d) \in [0,1]$ denotes the probability that a degree-d player chooses action 1. Here we focus on symmetric equilibria where players of the same degree follow the same strategy. Then $p_\sigma = \sum_d \sigma(d) P(d)$ is the probability that a neighbor takes action 1, the expected utility of degree-d player i given p_σ is

$$U_{d_i}(x_i, p_\sigma) = \sum_{0 \leq m \leq d_i} u_{d_i}(x_i, m) \begin{pmatrix} d_i \\ m \end{pmatrix} p_\sigma^m (1 - p_\sigma)^{d_i - m}$$

Another assumption is $u_d(x_i, m) = u_{d+1}(x_i, m)$, means that given action x_i, payoffs only depend on absolute numbers of neighbors taking action 1.

Definition 2.9 (BNE)

A strategy profile σ is a mixed strategy symmetric Bayesian Nash equilibrium if

1. $U_d(1, p_\sigma) \geq U_d(0, p_\sigma)$ *if* $\sigma(d) > 0$, *and*
2. $U_d(1, p_\sigma) \leq U_d(0, p_\sigma)$ *if* $\sigma(d) < 1$.

Definition 2.10 (Strict strategic complement / substitute)

1. *A semi-anonymous graphical games shows strict strategic complements if* $u_d(1, m) - u_d(0, m) > u_d(1, m') - u_d(0, m')$ *for all d and* $m > m'$.
2. *A semi-anonymous graphical games shows strict strategic substitutes if* $u_d(1, m) - u_d(0, m) < u_d(1, m') - u_d(0, m')$ *for all d and* $m > m'$.

Lemma 2.7

1. *If it is a game of strict strategic complements, then all symmetric equilibria are nondecreasing in degree, i.e.* $\sigma(d+1) \geq \sigma(d)$.
2. *If it is a game of strict strategic substitutes, then all symmetric equilibria are nonincreasing in degree, i.e.* $\sigma(d+1) \leq \sigma(d)$.

Proof

■

Note on

1. *In the case of strict strategic complements, the equilibria is pure strategy:* $\sigma^t(d) = 1$ *for* $d \geq t$ *and* $\sigma^t(d) = 0$ *for* $d < t$.
2. *In the case of strict strategic substitutes, mixing may occur.*

Theorem 2.1

1. *Consider a game of strict strategic complements that has an equilibrium with threshold* t. *If the degree distribution is changed to* P' *such that* $\sum_{d \leq t} P'(d) \leq \sum_{d \leq t-1} P(d)$, *then there is an equilibrium threshold under* P' *that is at least as low as* t.
2. *Consider a game of strict strategic substitutes that has an equilibrium with threshold* t. *If the degree distribution is changed to* P' *such that* $\sum_{d \leq t} P'(d) \geq \sum_{d \leq t-1} P(d)$, *then there is an equilibrium threshold under* P' *that is at least as high as* t.

3 Game-Theoretic Network Formation

3.1 Strong Stability

In this section, we first start by two games to discuss some drawbacks of stability, and provides another definition of stability.

3.1.1 Extensive form game of Network Formation (Aumann and Myerson, 1988)

To be continue

3.1.2 Simultaneous Link-Announcement Game (Myerson, 1977, Graphs and Cooperation in Games)

Suppose n players simultaneously announce a subset of players as friends (excluding herself), and i's set of strategies is $S_i = 2^{N \setminus \{i\}}$.[2] For each strategy profile $s \in S_1 \times \cdots \times S_n$, link ij is formed iff $j \in s_i$ and $i \in s_j$. The resulting network given s is $G(s) = \{ij \mid j \in s_i \text{ and } i \in s_j\}$. Each player's utility $u_i(G)$ is a funciton of network G.

Definition 3.1 (Nash Stable)

G is Nash stable if there is a strategy profile s that is a Nash equilibrium of the game and $G = G(s)$.

[2] w^A is a notation of the power set of A, which refers to the set of all subsets of A.

Lemma 3.1

Network G is Nash stable iff no player wishes to delete a set of his or her links, i.e., there is no subset $H \subseteq G$ of i's neighbors such that

$$u_i(G - \{ik : k \in H\}) > u_i(G)$$

Proof

■

Note on *Problems of Nash Stability*

1. *Nash stability does not capture the fact that it may be mutually beneficial for two players to form a new relationship. For example, $s_i = \emptyset$ for all i is always a Nash equilibrium, regardless of the payoffs. Each player refuses to link with any other player, because he or she correctly forecasts that the other players will do the same.*
2. *Players may play weakly dominated strategies in equilibrium. Can we use a refinement of Nash equilibrium where players do not play weakly dominated strategies simply? The answer is no. For example, all strategies in this link-announcement game are undominated[3], and the empty network is an outcome of a Nash equilibrium that only uses undominated strategies, obviously this is not a reasonable prediction we want.*

Note on *Problems of Pairwise Stability Pairwise stability overcomes the difficulties inherent in Nash stability, but it does not allow for cutting or creating multiple links at one time, and this may lead to over-connected networks being pairwise stable.*

Definition 3.2 (Pairwise Nash Stable (Jackson and Wolinksy, 1996, A strategic model of social and econon

G is pairwise Nash stable if it is both Nash stable and pairwise stable.

Note on *Existence and convergence Consider a sequence of networks that might arise as players add and delete links to improve their payoffs, if the sequence converges, the limit network will be stable. Thus existence can be interpreted as the convergence of sequence of networks.*

Definition 3.3 (Adjacent network)

Two networks G and G' are adjacent if they differ by only one link, i.e., $G' = G + ij$ for some $ij \notin G$ or $G' = G - ij$ for some $ij \in G$.

Definition 3.4 (Defeat)

Network G' defeats an adjacent network G if either

1. *$G' = G - ij$ and $u_i(G') > u_i(G)$, or*
2. *$G' = G + ij$ and $u_i(G') \geq u_i(G)$ and $u_j(G') \geq u_j(G)$, with at least one inequality holding strictly.*

[3] The strategy space $s_1 = \{\emptyset, \{s_2\}, \{s_3\}, \{s_2, s_3\}\}$, for each strategy, we can show it is undominated by other strategies.

Lemma 3.2

Network G is pairwise stable iff it is not defeated by any adjacent networks.

Definition 3.5 (Improving path)

An improving path is a sequence of distinct networks $\{G_1, G_2, \ldots, G_K\}$ such that each network G_k with $k < K$ is adjacent to and defeated by the subsequent network G_{k+1}.

Lemma 3.3

Network G is pairwise stable iff it has no improving path emanating from it.

Note on *If no pairwise stable network exists, there must exist at least one improving cycle, i.e., a sequence of adjacent networks $\{G_1, G_2, \ldots, G_K\}$ such that each network is defeated by the subsequent network G_{k+1} and $G_1 = G_k$.*

Note on *Example of nonexistence of PS networks*

Definition 3.6 (Ordinal potential function)

An ordinal potential function is a function $f : G(N) \to R$ such that G' defeats G iff $f(G') > f(G)$ and G' and G are adjacent.

Definition 3.7 (Potential game)

Games that admit ordinal potential functions are called potential games.

Note on *A common construction of a potential funciton is to set $f(G) = \sum_{i \in N} u_i(G)$.*

Theorem 3.1 (Existence)

If payoff funtions u admits an ordinal potential function, then there are no improving cycles. Conversely, when payoffs exhibit no indifference, there are no improving cycles only if there exists an ordinal potential function.

Proof

■

Note on *Payoff function profile u exhibit no indifference if for any two adjacent networks, one defeats the other.*

Note on *This theorem can be naturally extended to the case of pairwise Nash stability.*

Definition 3.8 (Weakly adjacent)

Two network G and G' are weakly adjacent if G' is obtained from G by adding a single link or by deleting some set of links such that each of the deleted links involves a same player.

Note on *Note that here we allow delete multiple links.*

Definition 3.9 (Obtainable)

G' is obtainable from G by player i if $g'_{kj} \neq g_{kj}$ implies that $k = i$: changes in the network involve links that are directed from i.

Definition 3.10 (Directed Nash stable)

G is directed Nash stable if $u_i(G) \geq u_i(G')$ for each i and all networks G' that are obtainable from G by player i.

Note on *Some applications might involve directed networks where links can be formed unilaterally.*

4 Learning on Networks

4.1 Contagion on Networks (Morris 2000 Restud)

In a n-player semi-anonymous graphical game with strategic complemetns, suppose action space is binary, where 1 means adopting a new technology. And action 1 is a best response iff at least fraction $q \in (0, 1)$ of the player's neighbors choose 1. There is at least a equilibria where all players choose the same action.

Definition 4.1 (r-cohesive)

A set S is said to be r-cohesive if each node in S has at least fraction r of its neighbors in S, that is, $\min_{i \in S} \frac{|N_i(G) \cap S|}{d_i(G)} \geq r$ for each $i \in S$.

Lemma 4.1

If S is r-cohesive, it is r'-cohesive for all $r' \leq r$.

Definition 4.2 (Cohesiveness)

Define cohesiveness of S as the maximum r such that S is r-cohesive.

Note on *Example*

Lemma 4.2

There is an equilibrium in which both actions are played iff there exists a nonempty subset of player S that is q-cohesive and its complement $\mathrm{N} \backslash S$ is $(1 - q)$-cohesive.

Proof Only if: Since we have a set S of m nodes with contagion eventually, and suppose its complement has some subset A that is more than $1 - q$-cohesive. Then the set of A of nodes will all play 0 at every step, and the contagion cannot occur eventually.

If: Since we have a set S of m nodes, and its complement is uniformly no more than $(1 - q)$-cohesive. This means that there must be at least one player in the complement who has at least a fraction of q of his or her neighbors in S. So, at the first step, at least these players changes strategies. Subsequently, at each step, we can find someone and he would like to change strategies. And so every player must eventually change. ■

Definition 4.3

There is contagion from m nodes if there is some set of m nodes whose initial infection leads to all players taking action 1.

Definition 4.4

A set S is said to be uniformly no more than r-cohesive if there is no nonempty subset of S that is more than r-cohesive.

Lemma 4.3 (Necessary and Sufficient condition for Contagion)

Contagion from m nodes occurs iff there exists a set of m nodes such that its complement is uniformly no more than $(1-q)$-cohesive.

Proof

■

4.2 Limited Bayesian Learning Model (Bala and Goyal 1998 Restud)

Bayesian updating is a bit cumbersome to use in complex settings, nevertheless, this model provides useful insight, e.g., conditions under which individuals come to act similarly over time.

Suppose n players connected in an undirected connected network, agents simultaneously choose one of the two actions, A or B, in each period $t \in \{1, 2, ...\}$. The payoffs are random: A results in a payoff of 1 per period for sure, but B pays 2 with prob p and 0 with prob $1-p$. Each agent maximizes the expected sum of discounted payoffs: $E\left[\sum_t \delta^t \pi_{it}\right]$, where $\delta \in (0,1)$ is a discount parameter and π_{it} is the payoff that i receives at time t.

When p is known, B is optimal if $p > \frac{1}{2}$, while A is optimal if $p < \frac{1}{2}$. Suppose p is unknown and $p \in \{p_1, ..., p_K\}$ with $p_k \neq \frac{1}{2}$, and μ_i denotes a full-support prior over $\{p_1, \cdots, p_K\}$, where $\mu_i(p_k) > 0$ is the prob i initially assigns to p_k being the prob that action B pays 2.

Players can observe actions and outcomes chosen by his neighbors. Note that full bayesian learning is quite complicated: seeing that a neighbor chooses B may indicate that the neighbor's neighbors have had good outcomes from B in the past. Here we focus on limited byaesian learning where players only process the information from their neighbors' actions and outcomes, and ignore other direct information.

Lemma 4.4

With probability 1, there exists a time such that all agents end up in playing the same action from that time onward.

Proof

■

Lemma 4.5

For any $\varepsilon > 0$, there exists $\mu < 1$ such that if there exists at least one agent who has an initial belief greater than μ, then the prob that all agents eventually choose the right action is at least $1 - \varepsilon$.

Note on *Another example: HW2.3*

4.3 DeGroot Model (Degroot 1974 JASA; Golub and Jackson 2010 AEJ Micro)

n players start with initial opinions on a subject, and their initial belief denotes the probability that a given statement is correct, e.g. $p(0) = (p_1(0), \cdots, p_n(0)) \in [0, 1]$. A possibly weighted and directed $n \times n$ nonnegative matrix T describes the interaction patterns among players, where T_{ij} is the trust i puts on j's belief in forming i's belief for the next period. The sum of (row) trust is 1, so that T is a row stochastic matrix. The belief update process thus can be explained as $p(t) = Tp(t-1) = T^t p(0)$.

Note on *Example For example, in this updating matrix, agent 1 weights all beliefs equally. Suppose $p(0) = (1, 0, 0)'$, then $p(1) = Tp(0) = (\frac{1}{3}, \frac{1}{2}, 0)'$, and $p(t) \to (\frac{3}{11}, \frac{3}{11}, \frac{3}{11})'$.*

$$T = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/2 & 1/2 & 0 \\ 0 & 1/4 & 3/4 \end{pmatrix}$$

Note on *Incorporating media and opinion leaders Fixed sources of information can be simply viewed as i's with $T_{ii} = 1$ and $T_{ij} = 0$, that is, an external source of information is modeled as an agent i whose opinion stays fixed at $p_i(0)$, but whom other nodes pay attention to.*

Definition 4.5

T is said to be strongly connected if each node has a directed path to any other nodes.

Definition 4.6

A closed set of agents is $C \subseteq \{1, \ldots, n\}$ such that there is no directed path from a player in C to any player outside C, that is, there is no pair $i \in C$, $j \notin C$ such that $T_{ij} > 0$.

Definition 4.7

T is aperiodic if the greatest common divisor of all the directed cycle lengths is one.

Theorem 4.1

T is convergent iff every set of nodes that is strongly connected and closed is aperiodic.

Note on *Example*

Definition 4.8 (Consensus)

A group of players C reaches a consensus for an initial vector of beliefs $p(t)$ if

$$\lim_t p_i(t) = \lim_t p_j(t) \text{ for each } i, j \in C.$$

Lemma 4.6

Any strongly connected and closed group of players reaches a consensus iff it is aperiodic. Thus, they reach consensus when T converges.

Proof

■

Corollary 4.1

A consensus is reached in the Degroot model iff there is exactly one strongly connected and closed group of agents and T is aperiodic on that group.

Proof If there is more than one strongly connected and closed group, they may reach different consensus among each group. ■

Corollary 4.2

A consensus is reached in the DeGroot model iff there exists t such that some column of T^t has all positive entries.

Proof Note that ijth entries of T^t actually denotes i's trust on j's belief after t period or t paths. Some column of T^t being all positive actually means that all agents have an indirect path to these agents (column), and thus there exists one closed and strongly connected group. Note that even after t periods, these columns hold positive. Since all entries of some columns of T are nonnegative, thus for T^{t+1}, new columns' entries are actually conical combination of old columns' positive entries, thus remain positive. And this means that there are always indirect path of length $t+1, t+2, ...$, which ensures the greatest common divisor is one, and that is, aperiodicity. ■

Definition 4.9

Given that T is convergent and $p(\infty) = (p^\infty = \cdots = p^\infty)$ is the limiting consensus belief, an influence (row) vector $S \in [0,1]^n$ is defined by $p^\infty = s \cdot p(0) = \sum s_i p_i(0)$, where $\sum s_i = 1$. Since $s \cdot p(1) = s \cdot p(0)$, we have $s \cdot (\mathrm{Tp}(0)) = s \cdot p(0)$ for all $p(0)$, and hence $sT = s$ and s is a left-hand unit eigenvector of T.

Note on

- *The limiting consensus is weighted averages of the initial beliefs and the influences gives the relative weights.*
- *Since $S_j = \Sigma_i T_{ij} S_i$, a player acquires influence from being pointed by players who are themselves influential.*

- *If player j receives more weight than player k and $T_{ij} \geq T_{ik}$ for all i, then j has more influence than k.*

Note on *When T satisfies the regularity conditions, such s exists and is unique.*

Definition 4.10

Consider a sequence of societies $(T^n, p^n(0))$ with population n, suppose the true state of nature is μ, and $p_i^n(0) = \mu + e_i$ is a random variable with mean μ. The sequence of networks $(T_n)_{n=1}^{\infty}$ is wise if for every $\varepsilon > 0$,

$$\lim_n \Pr\left[|p^n(\infty) - \mu| \geq \varepsilon\right] = 0.$$

Note on

- *Since $p^n(\infty) = \sum_i s_i^n p_i^n(0)$, as long as we can invoke the law of large numbers and each player has the same influence, the sequence of networks is wise.*
- *More generally, as long as no agent has too much influence, the sequence of networks will be wise.*

Lemma 4.7

Under the DeGroot model, suppose the network is strongly connected and aperiodic,

1. *show that if $T_{ji} = T_{ij}$ for all i, then every agent has the same influence;*
2. *more generally, show that if $\sum_j T_{ji} = 1$ for all i, then every agent has the same influence.*

Proof Note that strong connection and aperiodicity ensure that there is a unique solution to $sT = s$. And the same influence means that $s_i = \frac{1}{n}$ for all i. More specifically, if $\sum_j T_{ji} = 1$ for all i, then obviously $s_i = \frac{1}{n}$ is a solution to $s_i = \sum_j T_{ji} s_j$. ■